Chapter 2 Exact Solutions

incompressible, viscous flow: $\begin{cases} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial P/\rho}{\partial x_i} + \nu \nabla^2 u_i + f_i \\ \frac{\partial u_i}{\partial x_i} = 0 \end{cases}$ $\Rightarrow$ pressure is simply a force to enforce mass conservation $\rho\nu = \mu$

2.1 Steady fully-developed duct flow

mass: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow$ automatically satisfied

x-mom.: $\cancel{\frac{\partial u}{\partial t}} + u\cancel{\frac{\partial u}{\partial x}} + \cancel{v}\frac{\partial u}{\partial y} + \cancel{w}\frac{\partial u}{\partial z} = -\frac{\partial P/\rho}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

$\Rightarrow \mu \nabla_\perp^2 u = \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{dp}{dx} = \text{const} \Rightarrow \nabla_\perp^2 u = \frac{1}{\mu}\frac{dp}{dx} = -C$

$\Rightarrow$ no-slip B.C.: $u = U_{wall}$

$\Rightarrow$ mass flow rate: $\dot{m} = \iint_{A_\perp} \rho u \, dS \Rightarrow$ bulk velocity: $U_{bulk} = \frac{1}{A_\perp}\iint_{A_\perp} u \, dS \Rightarrow \dot{m} = \rho U_{bulk} A_\perp$

$\Rightarrow$ wall shear stress: $\tau_{wall} = -\mu \frac{\partial u}{\partial n}$ (注意结合 $\tau_{ij} n_j = 2\mu S_{ij} n_j$)

$\Rightarrow$ integral analysis: only pressure and shear remain: $\overline{\tau}_{wall} = \frac{1}{P}\int_P \tau_{wall} \, dS$

$\overline{\tau}_{wall} P dx = -dp A_\perp \Rightarrow \frac{dp}{dx} = \frac{\overline{\tau}_{wall} P}{A_\perp}$ $d_h = \frac{4A_\perp}{P} = 2R$ (circle) $\Rightarrow \frac{dp}{dx} = -\frac{4\overline{\tau}_{wall}}{d_h}$

$\Rightarrow$ friction factor: relates flow rate to pressure drop / shear

$f_{DW} = \frac{-\frac{dp}{dx} d_h}{\frac{1}{2}\rho U_{bulk}^2}$, $f_F = \frac{\overline{\tau}_{wall}}{\frac{1}{2}\rho U_{bulk}^2} = \frac{1}{4} f_{DW}$

2.2 Poiseuille channel flow $\Rightarrow$ driven by dp/dx, $U_{wall} = 0$

x-mom.: $\frac{\partial^2 u}{\partial y^2} = -C = \frac{1}{\mu}\frac{dp}{dx}$, $u(y=\pm h) = 0 \Rightarrow u(y) = \frac{1}{2}C(h^2 - y^2) = -\frac{1}{2}\frac{1}{\mu}\frac{dp}{dx}(h^2 - y^2)$

2.3 Couette channel flow $\Rightarrow$ driven by relative boundary movement

x-mom.: $\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx} = 0$, $u(-h) = 0$, $u(h) = U$

$\Rightarrow u(y) = U\left(\frac{1}{2} + \frac{y}{2h}\right)$. $\tau_{wall} = \mu u'(y) = U/2h = \text{const.}$

2.4 Stokes' first problem: flow dragged by an impulsively started flat plate

- potential flow: $\vec{\omega} = \nabla \times \vec{u} = 0$, $\vec{u} = \nabla\phi$, since $\nabla\cdot\vec{u} = 0 \Rightarrow \nabla^2\phi = 0 \Rightarrow$ no viscosity body force

$\Rightarrow$ only require normal wall velocity $= 0$. can't guarantee no-slip

$\Rightarrow$ Simplest irrotational flow: $u(t=0) = (U, 0, 0)$ B.C. $u(y=0, t) = 0$ is imposed at $t=0$

$\Rightarrow$ x-mom.: $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$

$\Rightarrow$ Similarity variable: $\eta = \frac{y}{\delta(t)}$. $f(\eta) = \frac{u(y,t)}{U}$, B.C.: $f(\eta=0) = 0$, $\delta(t=0) = 0$, $f(\eta\to\infty) = 1$

$\Rightarrow f''(\eta) = -\frac{\delta}{\nu}\frac{d\delta}{dt}\eta f'(\eta) \Rightarrow \frac{\delta}{\nu}\frac{d\delta}{dt} = C = \text{const.}$ $(\delta(0) = 0)$ $\boxed{\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-x^2)\,dx}$

$\Rightarrow \delta = \sqrt{2C\nu t}$, $\eta = \frac{y}{\sqrt{2C\nu t}}$, $f''(\eta) = -C\eta f'(\eta)$, $(f(0) = 0,\ f(\infty) = 1)$

$\Rightarrow f(\eta) = \mathrm{erf}\left(\sqrt{\frac{C}{2}}\eta\right)$, $u = Uf(\eta) = U\cdot\mathrm{erf}\left(\sqrt{\frac{C}{2}}\eta\right) = U\cdot\mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)$

$\Rightarrow \tau_w = \mu\frac{\partial u}{\partial y}\Big|_{y=0} = \rho U\sqrt{\frac{\nu}{\pi t}} \Rightarrow$ vorticity $\omega_z(y) = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right)$

$\Rightarrow$ change reference frame: fluid at rest dragged by a plate at $-U$:

$\Rightarrow u(y,t) = -U\left[1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]$

2.5 Stokes' second problem: flow dragged by an oscillating flat plate: $\omega = 2\pi f$

B.C. $u(y\to\infty)=0$, $u(y=0)=U(t)=U\cos\omega t=Re\{Ue^{i\omega t}\}$

$\Rightarrow$ Fourier transform: $t\to\omega'$ $\Rightarrow$ finally get: $u(y,t)=U\exp\left(-y\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(\omega t-y\sqrt{\frac{\omega}{2\nu}}\right)$

$\Rightarrow$ thickness of viscous effected region: $\delta\sim\sqrt{\frac{2\nu}{\omega}}$

$\Rightarrow$ at $y^*=\pi\sqrt{\frac{2\nu}{\omega}}$, overshoot, $u(y^*)=-1.04U\cos\omega t$

$\Rightarrow \tau_w=-U\sqrt{\frac{\omega}{\nu}}\cos(\omega t+\frac{\pi}{4})$ $\Rightarrow$ max τ_{wall} arrives earlier than max velocity

2.6 General solution to unsteady flow above a flat plate $u(0)=U(t)$, $u(\infty)=0$

$$u(y,t)=\int_{-\infty}^{t}\frac{dU}{dt}\Big|_{t=\tau}\left[1-\operatorname{erf}\left(\frac{y}{2\sqrt{\nu(t-\tau)}}\right)\right]d\tau$$

$\Rightarrow$ For Stokes' 1st problem, $\frac{dU}{dt}=U\delta(t)$ $H(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}\to\frac{dH}{dt}=\delta(t)\to\int_{-\infty}^{\infty}f(t)\delta(t)\,dt=f(0)$

$\Rightarrow \tau_w=-\rho\sqrt{\nu}\int_{-\infty}^{t}\frac{\frac{dU}{dt}\big|_{t-\tau}}{\sqrt{\pi(t-\tau)}}d\tau$ $\Rightarrow$ a slowly decaying integral of all past accelerations

Caputo fractional derivative:

$$\frac{d^{\frac{1}{2}}U}{dt^{\frac{1}{2}}}=\int^{t}\frac{dU/dt}{\sqrt{\pi(t-\tau)}}d\tau$$

2.7 Ekman layer

flow homogeneous in x-y surface, τ_w fixed

$\mu\frac{\partial u_1}{\partial x_3}=\tau_w$, $\frac{\partial u_2}{\partial x_3}=0$, $u_3=0$

$$\boxed{u_1(z)=U\exp\left(\frac{z}{\delta}\right)\cos\left(\frac{z}{\delta}-\frac{\pi}{4}\right)}$$

$$\boxed{u_2(z)=U\exp\left(\frac{z}{\delta}\right)\sin\left(\frac{z}{\delta}-\frac{\pi}{4}\right)}$$

$\Rightarrow \begin{cases}u_1(0)=\frac{U}{\sqrt{2}}\\ u_2(0)=\frac{-U}{\sqrt{2}}\end{cases}$

net transport orthogonal to wind stress

Chapter 3 Approximate solutions for $Re \ll 1$

$Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu} = \frac{慣性}{黏性} \Rightarrow Re \ll 1 \Rightarrow$ 慣性可忽略

$\rho \nu = \mu$

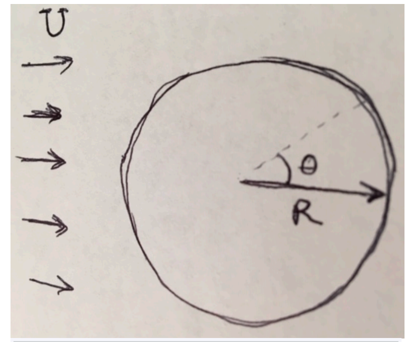
3.1 Creeping flow equations

$$Re\left(\frac{\partial u_i^*}{\partial t^*} + u_i^* \frac{\partial u_i^*}{\partial x_j^*}\right) = -\frac{\Delta P L}{\mu U}\frac{\partial p^*}{\partial x_i^*} + \frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} \Rightarrow \Delta p = \frac{\mu U}{L} \Rightarrow \mu \nabla^2 \vec{u} = \nabla p - \vec{f}$$

Drag: $\frac{F_D}{\mu U L}$ = function of geometry only $\Rightarrow C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L} = \frac{F_D/\mu U L}{\frac{1}{2}\frac{\rho U}{\mu}} \sim \frac{1}{Re_L}$

$\Rightarrow$ take curl: $\mu \nabla^2 \vec{\omega} = \nabla \times \vec{f} \rightarrow$ viscosity diffuse to vorticity

3.2 Stokes flow past a sphere

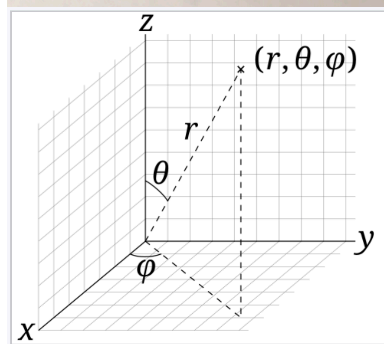


define a streamfunction: $\psi \Rightarrow u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$

and write vortex: $\omega_\phi = \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \qquad \frac{\partial}{\partial \phi} = 0$

$\Rightarrow$ plug into $\nabla^2 \vec{\omega} = 0$ (momentum conservation)

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) - \frac{1}{r^2 \sin^2\theta}\right]\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\right)\right]\psi = 0$$



The ***physics convention***. Spherical coordinates (r, θ, φ) as commonly used: (ISO 80000-2:2019): radial distance r (slant distance to origin), polar angle θ (theta) (angle with respect to positive polar axis), and azimuthal angle φ (phi) (angle of rotation from the initial meridian plane).

$\Rightarrow$ B.C. $u_r(r=R,\theta) = 0 \Rightarrow \psi$ = const on surface, no penetration

$u_\theta(r=R,\theta) = 0 \Rightarrow$ no slip

$u_r(r\to\infty,\theta) = U\cos\theta \qquad u_\theta(r\to\infty,\theta) = -U\sin\theta$

$$\Rightarrow \boxed{\psi(r,\theta) = \frac{1}{2}Ur^2\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)\sin^2\theta} \Rightarrow \boxed{u_r(r,\theta) = U\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)} \quad \boxed{u_\theta(r,\theta) = -U\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)}$$

$\Rightarrow$ this is an exact solution for Stokes' equation

$\Rightarrow$ asymptotically correct for N-S with $Re = \frac{UD}{\nu} \to 0$ at finite Re, this solution has finite error in satisfing N-S.

Compare with irrotational flow: $\vec{\omega} = 0$ (no no-slip)

$$\boxed{\psi(r,\theta) = \frac{1}{2}Ur^2\left(1 - \frac{R^3}{r^3}\right)\sin^2\theta} \quad \boxed{u_r(r,\theta) = U\cos\theta\left(1 - \frac{R^3}{r^3}\right)} \quad \boxed{u_\theta(r,\theta) = -U\sin\theta\left(1 + \frac{R^3}{2r^3}\right)}$$

3.3 Stokes drag on a sphere

$$\nabla p = \mu \nabla^2 \vec{u} \Rightarrow \boxed{p(r,\theta) = p_\infty - \frac{3\mu U R\cos\theta}{2r^2}}$$

$\Rightarrow$ Stokes' drag law:

① pressure drag: $F_{D,p} = -\int_0^{2\pi}\int_0^{\pi} p(r,\theta)\cos\theta\,(R^2\sin\theta\, d\theta\, d\phi) = 2\pi\mu U R$

② Viscous drag: $F_{D,\mu} = -\int_0^{2\pi}\int_0^{\pi} \mu \frac{\partial u_\theta}{\partial r}\Big|_{R,\theta} \sin\theta\,(R^2 \sin\theta\, d\theta\, d\phi) = 4\pi\mu U R$

$\Rightarrow$ total drag: $F_D = 6\pi\mu U R \quad (Re_D \lesssim 1)$

Btw. compare to irrotational flow:

$$\boxed{\frac{p - p_\infty}{\frac{1}{2}\rho U^2} = \left(3\cos^2\theta - 1\right)\frac{R^3}{r^3} - \left(3\cos^2\theta + 1\right)\frac{R^6}{4r^6}}$$

3.4 Small spherical particle in an arbitrary flow

$$m_p\frac{d\mathbf{v}}{dt} = \underbrace{(m_p-m_f)\mathbf{g}}_{F_1} + \underbrace{m_f\frac{D\mathbf{u}}{Dt}}_{F_2} - \underbrace{\frac{1}{2}m_f\left(\frac{d\mathbf{v}}{dt}-\frac{D\mathbf{u}}{Dt}\right)}_{F_3} - \underbrace{3\pi\mu d_p(\mathbf{v}-\mathbf{u})}_{F_4} - \underbrace{\frac{3}{2}\pi\mu d_p^2\int_0^t d\tau\frac{\frac{d}{d\tau}(\mathbf{v}-\mathbf{u})}{\sqrt{\pi\nu(t-\tau)}}}_{F_5}$$

3.5 Stokes drag on drops / bubbles (no longer no-slip at surface) $\Rightarrow F_D = 4\pi\mu_o UR\dfrac{\mu_o+\mu_i}{\mu_o+\frac{2}{3}\mu_i}$

3.6 Stokes drag on an arbitrarily shaped object

F_D (largest sphere that fits inside it) $\leq F_D \leq F_D$ (smallest sphere into which it fits)

Also, $F_D = C(\text{shape, orientation})\,\mu U L$

$\Rightarrow$ Stokeslet: far away from object, flow doesn't care shape

$\mu\nabla^2\vec{u} = \nabla p - \vec{F}\delta(\vec{x})$ Oseen-Burgers tensor

$$\boxed{\mathbf{u}^f = \frac{\mathbf{F}}{4\pi\mu|\mathbf{x}|}} \quad \boxed{u_i^p = -\frac{F_j}{8\pi\mu|\mathbf{x}|}\left(\delta_{ij}-\frac{x_ix_j}{|\mathbf{x}|^2}\right)} \Rightarrow \boxed{u_i = \frac{1}{8\pi\mu}G_{ij}F_j} \qquad G_{ij} = \frac{1}{|\mathbf{x}|}\left(\delta_{ij}+\frac{x_ix_j}{|\mathbf{x}|^2}\right)$$

3.7 Oseen correction

Compare $\nabla p = \mu\nabla^2\vec{u}$ with $\rho\vec{u}\cdot\nabla\vec{u} = -\nabla p + \mu\nabla^2\vec{u}$ (finite Re)

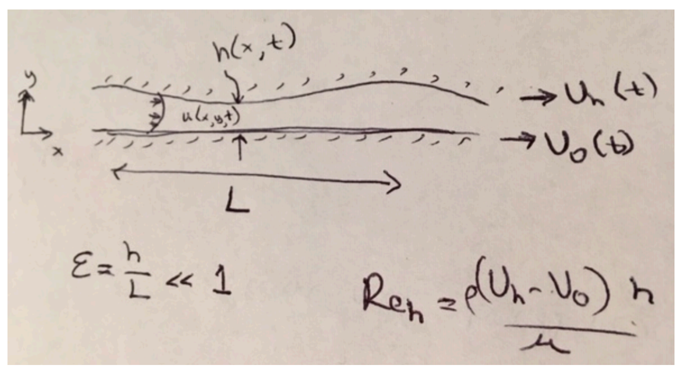
$\Rightarrow$ total relative error: $\varepsilon = \dfrac{|\rho\mathbf{u}\cdot\nabla\mathbf{u}|}{|\nabla p|} \sim \dfrac{\rho U^2R/r^2}{\mu UR/r^3} = \dfrac{\rho U r}{\mu} = Re_r = Re_D\dfrac{r}{D}$

$\Rightarrow$ Oseen radius: outside which error is significant $\Rightarrow r^* \sim \dfrac{\mu}{\rho U}$

$\Rightarrow$ Oseen's approximation:

$\vec{u}\cdot\nabla\vec{u} = \vec{U}\cdot\nabla\vec{u} + \cancelto{0}{(\vec{u}-\vec{U})}\cdot\nabla\vec{u} \Rightarrow$ new eq: $\rho\vec{U}\cdot\nabla\vec{u} = -\nabla p + \mu\nabla^2\vec{u} \Rightarrow$ analytical solution

$\Rightarrow F_D = 3\pi\mu U d_p\left(1+\frac{3}{16}Re\right)$, $Re = \dfrac{U d_p}{\nu}$



3.8 Lubrication flows

$\Rightarrow$ dimentional analysis: (mass Eq)

$v^* = \dfrac{v}{\varepsilon U}$ (v is 1-order smaller than u)

x-mom. $\varepsilon Re_h\left(\dfrac{\partial u^*}{\partial t^*}+u^*\dfrac{\partial u^*}{\partial x^*}+v^*\dfrac{\partial u^*}{\partial y^*}\right) = -\dfrac{\Delta P h^2}{\mu U L}\dfrac{\partial p^*}{\partial x^*}+\varepsilon^2\dfrac{\partial^2 u^*}{\partial x^{*2}}+\dfrac{\partial^2 u^*}{\partial y^{*2}}$

$\varepsilon = \dfrac{h}{L}$, $\quad Re_h = \dfrac{\rho U h}{\mu}$ $\quad \Rightarrow \Delta p = \mu U L/h^2$

y-mom. $\Rightarrow -\partial p^*/\partial y^* = 0$

$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{\mu}\dfrac{\partial p}{\partial x}\end{cases} \Rightarrow u(x,y,t)$ in terms of $\dfrac{\partial p}{\partial x} \Rightarrow$ solve for $\dfrac{\partial p}{\partial x}$

$$\boxed{u(x,y,t) = -\frac{h^2(x,t)}{2\mu}\frac{\partial p}{\partial x}\left(1-\frac{y}{h(x,t)}\right)\frac{y}{h(x,t)}+[U_h(t)-U_0(t)]\frac{y}{h(x,t)}+U_0(t)} \qquad \boxed{\frac{\partial Q}{\partial x} = -\frac{\partial h}{\partial t}}$$

$\Rightarrow \dfrac{\partial p}{\partial x} = \dfrac{6\mu}{h^2}(U_h+U_0) - \dfrac{12\mu Q}{h^3}$ where $Q \equiv \int_0^{h(x,t)} u(x,y,t)\,dt \Rightarrow$ just to present $\frac{\partial p}{\partial x}$ in terms of $Q(x)$

$\Rightarrow$ mass Eq.: $Q = Q_0 - \int_0^x \left(U_h\frac{\partial h}{\partial x} - v(h)+v(0)\right)dx \Rightarrow$ get $Q(x) \Rightarrow$ get $p \Rightarrow$ get u

$\Rightarrow$ lubrication force upward force: $F = \int_0^L p(x)\,dx$

we like fix $h_L \rightarrow$ at $h_L/h_0 = 0.457$, $F \rightarrow \max$

$\Rightarrow$ Reverse flow: $\left.\frac{\partial u}{\partial y}\right|_{y=h(x)} > 0 \Rightarrow h(x) > \frac{3Q}{U} \Rightarrow h_L/h_0 < \frac{1}{2} \rightarrow$ at h_0

$\Rightarrow$ overshoot: $\left.\frac{\partial u}{\partial y}\right|_{y=0} < 0 \Rightarrow h(x) < \frac{3Q}{2U} \Rightarrow h_L/h_0 < \frac{1}{2} \rightarrow$ at h_L

Chapter 4 Approximate solutions for $Re >> 1$ (page 90 – 142)

$\nu \nabla^2 \vec{u} = -\nu \nabla \times \vec{\omega}$

- 4.1 Inviscid flow:

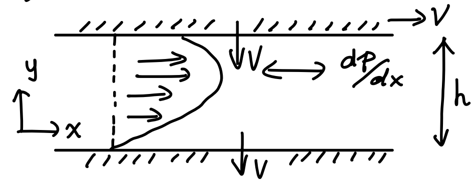
Potential flow: if $\nabla\times\vec{u}=0 \Rightarrow$ we have $\vec{u}=\nabla\phi \Rightarrow \begin{cases}\text{lift} & \checkmark \\ \text{drag} & \times \rightarrow \text{zero drag}\end{cases}$

and if $\nabla\cdot\vec{u}=0 \Rightarrow \nabla^2\phi=0$

dimensional scaling of incompressible momentum conservation: $\Delta p = \rho U^2$

$\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho}\nabla p$ Also written as $\frac{\partial\vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla\left(\frac{1}{2}|u|^2 + p/\rho\right) = 0$

4.2 Matched asymptotic expansions



Recall $\begin{cases}\text{Couette flow: plate moving} \\ \text{Poiseuille flow: pressure gradient driven}\end{cases} \Rightarrow \nu u_{yy} = -\frac{1}{\rho}\frac{dp}{dx}$

no slip: $u(0)=u(h)=0$

Mass: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow v = -V$ everywhere

y-momentum: $\frac{\partial p}{\partial y} = 0 \Rightarrow p = p(x)$

x-momentum: $\mu u_{yy} + \rho V u_y = \frac{dp}{dx}$

$\Rightarrow$ First, consider Exact solution: let $\varepsilon = \frac{1}{Re} \Rightarrow \varepsilon u'' + u' = A$, $u(0)=0$, $u(1)=1$

$\Rightarrow u(y) = \frac{1-A}{1-\exp(-\frac{1}{\varepsilon})}\left(1-\exp(-\frac{y}{\varepsilon})\right) + Ay$

$\Rightarrow$ Exact solution when $Re>>1$:

$u'(y) = \frac{1-A}{1-\exp(-\frac{1}{\varepsilon})}\frac{\exp(-\frac{y}{\varepsilon})}{\varepsilon} + A \qquad u''(y) = \frac{-(1-A)}{1-\exp(-\frac{1}{\varepsilon})}\frac{\exp(-\frac{y}{\varepsilon})}{\varepsilon^2}$

Since $Re>>1 \Rightarrow \varepsilon = \frac{1}{Re} << 1 \Rightarrow \varepsilon << y < 1 \rightarrow u'' \sim O(\varepsilon) << u' \sim O(1)$

$\Rightarrow \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\exp(-\frac{y}{\varepsilon}) = 0 \Rightarrow u'(y) \approx A \Rightarrow$ this is exactly the inviscid solution

effective governing equation in the "outer" region: $u_o'(y) = A$

However if $y<<1$, $\varepsilon<<1$, but $y/\varepsilon = \text{const.} \Rightarrow \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\exp(-\frac{y}{\varepsilon}) = \infty \rightarrow u'' \sim u' \sim O(\frac{1}{\varepsilon}) >> A$

$\Rightarrow$ Governing equation for "inner" region: $\varepsilon u_I'' + u_I' = 0$, $0 < \frac{y}{\varepsilon} << \infty$

$\Rightarrow$ dimensionless: $u^* = \frac{u}{V}$, $y^* = \frac{y}{h}$

$\Rightarrow \frac{1}{Re}u^*_{y^*y^*} + u^*_{y^*} = A^* = \frac{h}{\rho V^2}\frac{dp}{dx} = \text{const}$ B.C. $\begin{cases}u^*(y^*=0) = 0 \\ u^*(y^*=1) = 1\end{cases}$

$\Rightarrow$ Introducing matched asymptotic expansion:

For "inner" equation, do rescaling: $\eta = \frac{y}{\delta(\varepsilon)} \Rightarrow \frac{d}{dy} = \frac{d\eta}{dy}\frac{d}{d\eta} = \frac{1}{\delta(\varepsilon)}\frac{d}{d\eta}$

$\Rightarrow \frac{\varepsilon}{\delta^2(\varepsilon)}\frac{d^2u_I}{d\eta^2} + \frac{1}{\delta(\varepsilon)}\frac{du_I}{d\eta} = A$

In the limit $\varepsilon\to 0 \Rightarrow \delta(\varepsilon) = \varepsilon$

$\Rightarrow \frac{d^2u_I^{(0)}}{d\eta^2} + \frac{du_I^{(0)}}{d\eta} = 0 \Rightarrow u_I^{(0)} = C_1 + C_2\exp(-\eta)$, $u_o^{(0)}(1) = 1$, $u_I^{(0)}(0) = 0$

Also $\lim_{\eta\to\infty} u_I(\eta) = \lim_{y\to 0} u_o(y)$, with $u_o^{(0)}(y) = Ay + (1-A)$

$\Rightarrow u_I^{(0)}(\eta) = (1-A)(1-\exp(-\eta))$

$\Rightarrow u(y) = (1-A)\left[1-\exp(\frac{-y}{\varepsilon})\right] + Ay$ composite solution

4.3 Boundary layer equations:

mass: after dimentionless: $\frac{U}{L}=\frac{V}{\delta} \Rightarrow V=\frac{\delta}{L}U$

Streamwise momentum: pressure term scaling: $(\Delta P)_x = \rho U^2$ → same as Bernoulli

Streamwise viscosity: $\frac{1}{Re}U_{xx} \to 0$

"inter-stream" viscosity scaling: $\frac{L^2}{\delta^2}\frac{1}{Re}=1 \Rightarrow$ B.L. thickness: $\delta(Re)=\frac{L}{\sqrt{Re}}$ → different to 4.2

$\Rightarrow u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-P_x+U_{yy}$

Wall-normal momentum: $(\Delta P)_y \sim \frac{1}{Re}(\Delta P)_x$

$\Rightarrow$ In B.L., $p(x,y)=P_\infty(x)$, which is determined by outer flow

wall-normal momentum itself can be neglected

$\Rightarrow$ outer flow: Bernoulli: $U_\infty\frac{dU_\infty}{dx}=-\frac{1}{\rho}\frac{dP_\infty}{dx}$

$\Rightarrow$ B.L. Eq.: $\begin{cases}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\\ u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_\infty\frac{dU_\infty}{dx}+\nu\frac{\partial^2 u}{\partial y^2}\end{cases}$ $\quad u(x,0)=0$, $u(x,\infty)=U_\infty$

4.4 Similarity Solution for B.L. flow over a plate

Stream function: $u=\frac{\partial\psi}{\partial y}$, $v=\frac{-\partial\psi}{\partial x}$ → plug to x-mom. ⇒ Continuity automatically satisfied

$$\Rightarrow \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U_\infty\frac{dU_\infty}{dx}+\nu\frac{\partial^3\psi}{\partial y^3},\quad \underbrace{\psi(x,y=0)=0}_{\text{no penetration}},\quad \overbrace{\left.\frac{\partial\psi}{\partial y}\right|_{x,y=0}=0}^{\text{no-slip}},\quad \text{and}\quad \lim_{y\to\infty}\frac{\partial\psi}{\partial y}=U_\infty(x)$$

replace: $\eta=\frac{y}{\delta(x)}$, $\phi(\eta)=\frac{\psi(x,y)}{U_\infty(x)\delta(x)} \Rightarrow \frac{\partial\eta}{\partial x}=-\frac{\eta}{\delta(x)}\frac{d\delta}{dx}$, $\frac{\partial\eta}{\partial y}=\frac{1}{\delta(x)}$ $\quad \frac{\partial\phi}{\partial\eta}=\frac{u}{U_\infty}$

$$\Rightarrow \overbrace{\underbrace{\boxed{\frac{x}{U_\infty(x)}\frac{dU_\infty}{dx}}}_{m}}^{\text{pressure}}\left[\phi'^2(\eta)-\phi(\eta)\phi''(\eta)-1\right]=\overbrace{\underbrace{\boxed{\frac{\nu x}{\delta^2(x)U_\infty(x)}}}_{C=1}}^{\text{viscous}}\phi'''(\eta)+\overbrace{\underbrace{\boxed{\frac{x}{\delta(x)}\frac{d\delta}{dx}}}_{\frac{1}{2}-\frac{1}{2}m}}^{\text{inertia}}\phi(\eta)\phi''(\eta).$$

$\Rightarrow$ viscous: $\frac{\nu x}{\delta^2 U_\infty}=C \Rightarrow \delta(x)=\sqrt{\frac{\nu x}{CU_\infty}}=\frac{x}{\sqrt{CRe_x}}$, $Re_x=\frac{U_\infty x}{\nu} \Rightarrow \eta=\frac{y}{\delta(x)}=y\sqrt{\frac{CU_\infty}{\nu x}}$

$\Rightarrow$ pressure: $\frac{x}{U_\infty}\frac{dU_\infty}{dx}=m \Rightarrow U_\infty(x)=Ax^m$

$\begin{cases}m<0: \text{adverse pressure gradient, APG}\\ m>0: \text{favorable pressure gradient, FPG}\end{cases}$

$\Rightarrow \frac{x}{\delta}\frac{d\delta}{dx}=\frac{1}{2}-\frac{1}{2}m$

$\Rightarrow$ Falkner-Skan $\phi'''(\eta)=m\left[\phi'^2(\eta)-1\right]-\frac{1}{2}(m+1)\phi(\eta)\phi''(\eta)$, $\boxed{\phi(0)=0,\quad \phi'(0)=0,\quad \lim_{\eta\to\infty}\phi'(\eta)=1}$

$\Rightarrow$ Assume the solution $\phi(\eta)$ is found $\eta(x,y)=y\sqrt{\frac{U_\infty(x)}{\nu x}}$ and $\frac{x}{U_\infty}\frac{dU_\infty}{dx}=m$ and $\frac{x}{\delta}\frac{d\delta}{dx}=\frac{1}{2}-\frac{m}{2}$

$\Rightarrow \boxed{u(x,y)=\frac{\partial\psi}{\partial y}=U_\infty(x)\phi'(\eta)}$ $\boxed{v(x,y)=\sqrt{\frac{\nu U_\infty}{x}}\left[\frac{1}{2}(1-m)\eta\phi'(\eta)-\frac{1}{2}(1+m)\phi(\eta)\right]}$

$\Rightarrow$ Now take $m=0 \Rightarrow U_\infty(x)=$ const $\Rightarrow$ Blasius $\phi'''(\eta)+\frac{1}{2}\phi(\eta)\phi''(\eta)=0$

$\Rightarrow$ B.L. growth: $\delta(x)\sim\sqrt{\frac{\nu x}{U_\infty}}$, $U_\infty\sim x^m \Rightarrow \delta(x)\sim x^{\frac{1}{2}}x^{-\frac{1}{2}m}=x^{\frac{1}{2}-\frac{1}{2}m}$

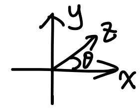
For both θ, δ, θ or δ (APG) > θ or δ (ZPG) ⇒ FPG thins B.L., increases C_f

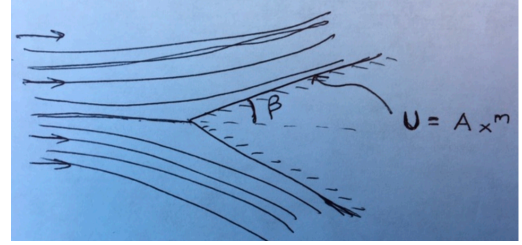
$\Rightarrow$ Skin friction: $\frac{\tau_w}{\rho}=\nu\left.\frac{\partial u}{\partial y}\right|_{y=0}=\nu\left.\frac{\partial^2\psi}{\partial y^2}\right|_{y=0}=\nu\frac{U_\infty}{\delta}\phi''(0)=\sqrt{\frac{\nu U_\infty^3}{x}}\phi''(0)\sim x^{(3m-1)/2}$

$\Rightarrow$ Skin friction coefficient: $C_f(x)=\frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}=\frac{2\phi''(0)}{\sqrt{Re_x}}\sim x^{-\frac{1}{2}(m+1)}$

$\Rightarrow$ Numerical solution to Blasius: $C_f=\frac{0.664}{\sqrt{Re_x}}$, $\delta_{99}=4.91\sqrt{\frac{\nu x}{U_\infty}}$

Flow over a wedge: $z = x + iy = re^{i\theta}$





define $F(z) = \phi(x,y) + i\psi(x,y)$

$\Rightarrow$ In 2D irrotational, $\nabla\cdot\vec{u} = 0$, $\nabla\times\vec{u} = 0$

$\Rightarrow u = \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x}$, $v = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y}$

$\Rightarrow$ "Complex" velocity $W(z) = \frac{dF}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u - iv$

$\Rightarrow$ If $F(z) = \frac{A}{m+1} z^{m+1} \Rightarrow W(z) = Az^m$

$\Rightarrow \psi = \frac{A}{m+1} r^{m+1} \sin((m+1)\theta) = 0 \Rightarrow \sin((m+1)\theta) = 0 \Rightarrow$ we choose $\theta = 0$, and $\theta = \frac{2\pi}{m+1}$

① $m = 0 \Rightarrow \theta = 0, 2\pi \Rightarrow$ uniform flow

② $m = \frac{1}{3} \Rightarrow \theta = 0$, $\theta = \frac{2\pi}{m+1} = \frac{3}{2}\pi \rightarrow \beta =$ "$-\frac{1}{2}\pi$" $\Rightarrow 2\beta = 2\pi - \frac{2\pi}{m+1} = \frac{2m\pi + 2\pi - 2\pi}{m+1} = \frac{2m\pi}{m+1}$

with $0 < m < 1$, along $\theta = 0 \Rightarrow$ FPG

③ $m = \frac{-1}{7} \Rightarrow \theta = 0$, $\theta = \frac{\pi}{m+1} = \frac{7}{6}\pi \Rightarrow$ APG

$\Rightarrow$ APG $\rightarrow C_f \downarrow \rightarrow$ B.L. grows faster than $\delta \sim \sqrt{x} \rightarrow$ seperation

4.6 momentum integral equation:

Plug mass equation into momentum equation:

$$\boxed{\frac{\partial[u(U_\infty - u)]}{\partial x} + \frac{\partial[v(U_\infty - u)]}{\partial y} + (U_\infty - u)\frac{dU_\infty}{dx} = -\nu\frac{\partial^2 u}{\partial y^2}} \Rightarrow \frac{d}{dx}\left[\int_0^\infty u(U_\infty - u)\,dy\right] + v(U_\infty - u)\Big|_{y=0}^{\infty} + \frac{dU_\infty}{dx}\left[\int_0^\infty (U_\infty - u)\,dy\right] = -\nu\frac{\partial u}{\partial y}\Big|_{y=0}^{\infty}$$

with $v(0) = 0$, $U_\infty = u(\infty) \Rightarrow$ $\boxed{\frac{d}{dx}\left[\int_0^\infty u(U_\infty - u)\,dy\right] + \frac{dU_\infty}{dx}\left[\int_0^\infty (U_\infty - u)\,dy\right] = \frac{\tau_w}{\rho}}$

$\Rightarrow$ displacement thickness: $\delta_*(x) = \int_0^\infty (1 - \frac{u}{U_\infty})\,dy$

momentum thickness: $\theta(x) = \int_0^\infty (1 - \frac{u}{U_\infty})\frac{u}{U_\infty}\,dy$

$\Rightarrow$ M.I.E.: $\frac{d(U_\infty^2\theta)}{dx} + \delta_* U_\infty \frac{dU_\infty}{dx} = \frac{\tau_w}{\rho}$ or $\frac{d\theta}{dx} + \frac{\delta_* + 2\theta}{U_\infty}\frac{dU_\infty}{dx} = \frac{\tau_w}{\rho U_\infty^2}$ (H = shape factor $= \frac{\delta_*}{\theta} \geq 1$)

since we have $C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} \Rightarrow \frac{d\theta}{dx} + \frac{\delta_* + 2\theta}{U_\infty}\frac{dU_\infty}{dx} = \frac{1}{2}C_f \Rightarrow \frac{d\theta}{dx} + (2+H)\frac{\theta}{U_\infty}\frac{dU_\infty}{dx} = \frac{1}{2}C_f$

If $U_\infty = $ const $\Rightarrow \theta(x) = \frac{1}{2}\int_0^x C_f(\xi)\,d\xi$

4.7 Integral approximation methods

$Re_\theta = \frac{U_\infty\theta}{\nu}$, $Re_{\delta_*} = \frac{U_\infty\delta_*}{\nu}$ $\Rightarrow$ need to determine H and C_f

$\Rightarrow \lambda = \frac{\theta^2}{\nu}\frac{dU_\infty}{dx}$, $C_f = (Re_\theta, \lambda)$, $H = (Re_\theta, \lambda) \Rightarrow \frac{d\theta}{dx} = \frac{1}{2}C_f - (2+H)\frac{\lambda}{Re_\theta}$

① Local self-similarity: $m = \frac{x}{U_\infty}\frac{dU_\infty}{dx}$ (outer pressure)

$\theta(x) = \alpha(m)\sqrt{\frac{\nu x}{U_\infty}} = \alpha(m)\,\delta(x) \Rightarrow \lambda = m\alpha^2(m)$

$\Rightarrow \frac{1}{2}C_f = \frac{\tau_w}{\rho U_\infty^2} = \frac{\alpha(m)\,\phi''(0)}{\sqrt{Re_\theta}}$ (no dependence on x)

$(2+H)\frac{\theta}{U_\infty}\frac{dU_\infty}{dx} = \frac{m\alpha(2\alpha + \beta)}{Re_\theta}$

$\Rightarrow \frac{d\theta}{dx} = \frac{f(m)}{Re_\theta} = \frac{F(x)}{Re_\theta} \Rightarrow \begin{cases} \text{Falkner-Skan: } A = 0.44,\ B = 5.3 \\ \text{Thwaites': } A = 0.45,\ B = 6.0 \end{cases}$

② Thwaites' method: $\frac{d\theta}{dx} = \frac{A - B\lambda}{2Re_\theta}$

$$\Rightarrow \boxed{\theta^2(x) = \left(\frac{U_\infty(0)}{U_\infty(x)}\right)^B \theta^2(0) + \frac{\nu A}{U_\infty^B(x)} \int_0^x U_\infty^{B-1}(\xi)d\xi} \Rightarrow \frac{1}{2}C_f = \frac{S(\lambda)}{Re_\theta} = \frac{(\lambda+0.09)^{0.62}}{Re_\theta}$$

③ VKP:

Set $\frac{u}{U_\infty} = f(\eta)$, $\eta = \frac{y}{\delta(x)}$ $\Rightarrow$ do curve fitting for $f(\eta)$

$f(0) = 0$ no slip $\quad f(1) = 1$ free stream

$f'(1) = 0$ $\quad f''(1) = 0$ $\quad f''(0) = -\Lambda = -\frac{\delta^2}{\nu}\frac{dU_\infty}{dx}$

continuous tress

$\Rightarrow f(\eta) = C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3 + C_4\eta^4$

$$\boxed{f(\eta) = F(\eta) + \Lambda G(\eta), \quad \text{where} \quad F(\eta) = 2\eta - 2\eta^3 + \eta^4, \quad \text{and} \quad G(\eta) = \frac{1}{6}\eta - \frac{1}{2}\eta^2 + \frac{1}{2}\eta^3 - \frac{1}{6}\eta^4}$$

$\Rightarrow \delta^*(x) = \int_0^\infty (1 - f(\eta))f(\eta)\frac{d\eta}{dy}dy = \int_0^1 (1-f)f\, d\eta \cdot \delta$

$\Rightarrow \frac{\delta^*}{\delta} = \frac{3}{10} - \frac{1}{120}\Lambda$

$\frac{\theta}{\delta} = \frac{37}{315} - \frac{1}{945}\Lambda - \frac{1}{9072}\Lambda^2$

Set $T = \frac{\delta^2}{\nu}$ $\Rightarrow$ replace $\frac{d\theta}{dx}$, $\frac{1}{2}C_f$, θ, δ^* with Λ

$\Rightarrow \frac{dT}{dx} = F_1(\Lambda)U_\infty^{-1} + F_2(\Lambda)\frac{d^2U_\infty}{dx^2}T^2$